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Contrasting and Looking into Some Mathematics Education Frameworks[1]

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Abstract: What are the fundamental features that characterize a research framework in mathematics education? What types of questions are important to ask in order to contrast and evaluate the potential associated with different frameworks? What vision of mathematics is endorsed or appears as important in particular perspectives? What types of tasks are used to promote learning within those perspectives? What instructional environments favor students learning under those frameworks? These questions were used as a guide to examine three important conceptual frameworks widely used in research and practice in mathematics education: Problem-solving, representations and visualization, and models-modeling perspectives.

Key words: Mathematics education frameworks; Research perspectives; Mathematical learning; Evaluation tool;

Introduction

Doing research in mathematics education involves the selection or construction and use of a research framework to support and guide the development of the inquiry process. However, as Lester (2005, p. 458) indicates "the notion of a research framework is central to every field of inquiry, but at the same time the development and use of frameworks may be the least understood aspect of the research process". An important aspect to evaluate research results in this field is to analyze the extent to which a set of principles associated with the framework embedded in the study is consistently utilized to reach and explain those results or findings. The existence of a diversity of frameworks to guide and direct research studies in the discipline makes necessary to review and contrast main tenets and principles that are

[1]The term framework is used to identify tenets, principles, assumptions, practices, and methods that support and guide research in mathematics education. Lester (2005) distinguishes three types of research frameworks: theoretical that rely on formal theories, practical that are based on accumulated practice knowledge of practitioners, previous research findings and views offered by public opinion, and conceptual frameworks that are seen as arguments in which the concepts chosen for investigation, and their relations are judged to be appropriate to support a research problem. Throughout the paper, we argue that frameworks used in research, in general, need to be examined and contrasted constantly and a tool to evaluate them needs to be developed. Thus, we use the term research perspective to identify a disciplined inquiry that embeds a particular framework that can be theoretical, conceptual or practical.

sometimes implicitly used to support and present research results. We do not intent to review extensively the variety of frameworks used in mathematics education research, this task would require a further work. Our goal in writing this paper is to point out to the research community in mathematics education about the importance of reviewing the main characteristics of the different frameworks. To start with, we have selected three particular frameworks to identify themes and questions that might help readers and researchers to examine and contrast important features associated with some of the existing frameworks. By doing this, we do not pretend to say that those frameworks cover all the areas in mathematics education. At the outset, it may be difficult to clearly identify research principles in published reports since they often do not provide enough information about the reasoning process used to reach research results.

Researchers usually present a polished product which may hide the ways in which projects got started, the insights on how they evolved, the hesitations, crossroads, decisions taken, the role of theory, and how when it shaped the research (Arcavi, 2000, p. 145).

In relation to the selection of a framework to support research projects, the editors of *Educational Studies in Mathematics* (*ESM*) (2002) suggest to "carry out comparative surveys of several theories, in particular of theories that purport to provide frameworks for dealing with the same related areas, topics and questions" (p. 253). How can one examine or contrast relevant research frameworks in mathematics education? Delving into the fundamentals of a theoretical perspective involves the consideration of an inquiry framework to orient the process of analysis. Thus, initial questions that can help organize and orient the analysis of a conceptual frame include: What does it mean to evaluate a research perspective in mathematics education? What are the main themes that any research frame needs to address or include? What type of questions do we need to pose and discuss in order to evaluate the potential of a particular research framework? How can we identify strengths and weaknesses in research studies that are endorsed by a specific theoretical frame? In particular, we are interested in discussing possible connections of some research perspectives with curriculum reforms, learning scenarios, and forms of evaluating students' mathematical competences. That is, the extent to which principles and concepts associated with those perspectives or research areas informs and supports instructional practices. In this context, Greeno, Collins, and Resnick (1996) stated that:

> …The role of theory in practice is not to prescribe a set of practices that should be followed, but rather to assist in clarifying alternative practices,

> including understanding of ways that aspects of practice related to alternative functions and purposes of activity (p. 40).

In the same vein, Sfard (2003) suggests that theories provide useful information to guide educational practice in different ways since each theory can support several curriculum orientations and practical instructional decisions:

> A theory, if well conceived, may lend support to a variety of educational practices without privileging any of them. Thus theory can only suggest, not dictate; curricular principles and concrete instructional approaches may be implied or supported by theory, but they are certainly not necessitated by theoretical arguments (Sfard, 2003, p. 354).

The dialectic nature of research and practice has been recognized in the educational community. Heid et al. (2006) state that "the practice of classroom mathematics teaching needs to be better informed by an understanding of the implications of existing bodies of research, and researchers need to learn more from the insights and knowledge of practitioners" (p. 76). Thus, it is important to reflect on the extent to which theoretical principles orient instructional practice to promote students' mathematical learning. In particular, the identification of the type of learning environment and vision of mathematics and learning that students are encouraged to develop under a perspective. As Schoenfeld (1992) stated: "goals for mathematics instruction depend on one's conceptualization of what mathematics is, and what it means to understand mathematics" (p. 334).

Delimitating the terrain or domain. Choosing themes to structure and organize information associated with each perspective necessarily reflects a position regarding what it might (or might not) be relevant to examine or look around the frameworks. Unfortunately, in mathematics education there has been a great interest in developing new conceptual frameworks and little work has been done around evaluating or testing the existing perspectives. Thus, the task itself of evaluating the perspectives by focusing on particular topics or questions seems to be an issue that needs to be part of the academic agenda of the discipline.

> [It] has become the norm rather than the exception for researchers to propose their own conceptual framework rather than adopting or refining an existing one in an explicit and disciplined way. This prolific theorizing …may also mean that theories are not sufficiently examined, tested, refined and expanded (*ESM* Editors, 2002, p. 253).

What are the most representative perspectives that have been used to frame mathematics education research? There may be different ways to respond to this question and we do not intend to address it directly. However, we chose three perspectives (problem solving, representations, and models and modeling) widely known in North-American research to introduce a tool that may be useful in examining and contrasting other perspectives. That is, we are not claiming that the perspectives we discuss are the most representative in mathematics education research. Rather, we take those perspectives as instances to introduce elements of a possible instrument that may be used to evaluate and contrast tenets or principles that guide research studies based on those perspectives. We argue that research agendas in mathematics education should include aspects related to the need to constantly evaluate research perspectives.

We present initially a general picture about what these three perspectives involve and later we examine the extent to which each perspective deals with a set of questions related to the nature of mathematical knowledge and learning, ways to describe and characterize mathematical competence, types of problems or learning activities that are important to promote students' learning, and ways to evaluate and communicate students' mathematical knowledge. We are interested in discussing what each perspective informs about students' mathematical competences and what aspects may be common or shared among those perspectives. In this context, we recognize that any conceptual framework or perspective constantly needs to be examined, refined or adjusted in terms of the development of the use of tools (particularly computational tools) that influences directly the ways students learn the discipline. At the end, we identify elements of an emerging framework that takes into account the use of dynamic representations in the process of developing and understanding students' mathematical ideas. In particular, we emphasize the use of this type of representation as a key element to reconstruct and develop mathematical ideas or results. Thus, students' learning is conceived as an ongoing and continuous process that is enhanced with the use of technological tools.

Elements of an Inquiry Process to Examine Research Frameworks

How can we recognize the existence of a particular theoretical framework? How is it constructed? Should any research report be embedded in a particular framework? What tools are important to evaluate strengths and limitations of that framework? What type of sources informs about the principles and basis associated with a particular framework? These questions reflect the kind of difficulties that might arise when one tries to examine closely the elements of a research perspective used to structure, organize, and guide research and practice in mathematics education. We recognize that the task itself of evaluating the robustness of a particular framework might focus on analyzing different issues and consequently take different directions. The focus, scope and direction of the analysis are determined by the themes to analyze, the questions to discuss, and the sources or material chosen to examine. Greeno, Collins, and Resnick (1996), for example, review and contrast three general perspectives about cognition and learning that have been developed in psychology research (empiricist, rationalist, and pragmatist-sociohistoric). They chose to focus on analyzing theoretical issues associated with each perspective related to the nature of knowledge, the nature of learning and transfer, and the nature of motivation and engagement. Regarding educational practices based on those perspectives, they discuss aspects related to the design of learning environments, analysis and formulation of curricula, and assessment.

In the same vein, Schoenfeld (2000) contrasts differences in using terms like theory or models in mathematics, science, and education and he identifies criteria to evaluate empirical or theoretical work in mathematics education. The criteria include: Descriptive power (what is important in the domain?), explanatory power (how and why things work), scope (what can it be covered?), predicting power (whether the theory can specify some results in advance), rigor and specificity (how well defined are the elements and relationships within the theory?),…, and multiple sources of evidence (use of different bodies of evidence to reach and explain the same result). To evaluate the extent to which a particular framework fulfills Schoenfeld's criteria demands the identification of features of the domain embedded in that framework. In general, a framework is known in terms of its uses or application and it seldom addresses directly aspects regarding its construction, nature, or development.

In this context, we organize our inquiry process by selecting a set of questions to examine and discuss issues related to the discipline, its practice and development; the process of learning it (what does it mean to learn mathematics); the students' participation in learning (how does learning take place); the problems or tasks used to promote learning; and evaluation of students' mathematics competences. The sources and materials that we chose to analyze represent seminal work associated with each perspective. However, we do not intent to do an extensive literature review of each perspective; instead we focus on analyzing what we judged are some representative sources. It is also important to mention that we have chosen three research perspectives to be examined, (problem-solving, representation and visualization, and model and modeling) since those are widely used to orient research and practice in mathematics education. However, we do not claim that these are the most representative within the discipline; but we emphasize the need to design an instrument to constantly evaluate the principles and methods associated with research perspectives in mathematics education.

At the beginning, we organize the discussion around fundamental themes that include: (a) features of mathematics knowledge (how can the discipline be characterized? What are the tools to develop and understand mathematics?); (b) learning environments: What does it mean to learn mathematics? What conditions favor students' learning? And (c) level of explanation of the process involved in students learning of new concepts: How do students enhance or construct mathematical knowledge beyond their current knowledge? These questions are discussed in terms of analyzing statements endorsed by each perspective. Later we contrast differences and similarities among them, and we argue about the need to readjust constantly tenets and principles around the research perspectives. What follows is the identification of main features associated with each perspective that were the basis to evaluate and contrast representative principles of each perspective.

Relevant Features Associated with Problem Solving, Representations and Visualization, and Model and Modeling Perspectives

We focus on discussing three research perspectives that are often utilized to support and guide the development of research studies in teaching and learning mathematics. Since there appear multiple interpretations about what each perspective may entail, it is convenient to identify main tenets and the sources used to analyze and contrast relevant aspects around each perspective. Thus, we start by describing relevant aspects associated with each perspective and later we identify the domain and type of questions that we discuss around those perspectives.

Problem Solving. Studies based on a problem solving approach (Schoenfeld, 1992, 1994) emphasize the importance for students to develop resources and strategies to think mathematically. Here problem-solving activities are crucial for students to learn and construct mathematical knowledge.

> Learning to think mathematically –involves a great deal more than having large amounts of subject-matter knowledge at one's fingertips. It includes being flexible and resourceful within the discipline, using one's knowledge efficiently, and understanding and accepting the tacit "rules of the game" (Schoenfeld, 1985, p. xii).

What does it mean to be flexible and resourceful in mathematics? What does it mean to use mathematical knowledge efficiently? What are the tacit "rules of the game" that one needs to understand and accept? How can one understand and accept those rules? These types of questions were part of the research programs in mathematics education that led to a recognition that a central component in developing students' mathematical thinking is that they need to acquire the habits, resources, strategies, and dispositions that reflect mathematical practice. That is, under this perspective there is a direct relationship between the process of doing or developing mathematics and the way students learn or construct mathematical knowledge.

> Learning to think mathematically means (a) developing a mathematical point of view –valuing the processes of mathematization and abstraction and having the prediction to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structures – mathematical sense making (Schoenfeld, 1994, p. 60).

Thus, the challenge for mathematics instruction seems to be how to create a mathematical microcosm that reflects the values of mathematical practices. But, what conditions are necessary for students and instructors to create that microcosm? Schoenfeld (1992) argues that:

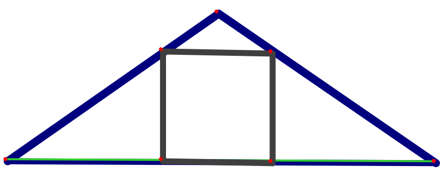
> …To develop the appropriate mathematical habits and dispositions of interpretations and sense-making as well as the appropriately mathematical modes of thought –then the community of practice in which they learn mathematics must reflect and support those ways of thinking. That is, classrooms must be communities in which mathematical sense-making, of the kind we hope to have students develop, is practiced (p. 345).

Thus, mathematics classrooms must reflect values and ways that are shown by mathematical communities when developing the discipline. Accordingly, an overarching conceptualization of mathematics that becomes relevant in problem solving is to think of mathematics as the science of patterns. Explicitly, Schoenfeld recognizes that a major shift in characterizing the nature of mathematics is to think of the discipline as the science of patterns. He cites statements from the National Research Council (NRC) to support this conceptualization:

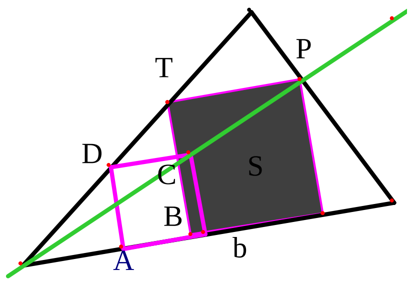
> Mathematics reveals hidden patterns that help us understand the world around us…. The process of “doing” mathematics is far more than just calculation or deduction; it involves observation of patterns, testing conjectures, and estimation of results (NRC, 1989, p. 31) (cited in Schoenfeld, 1992, p. 343).

In trying to categorize or explain students’ problem solving performance, there is a consensus about the existence of particular dimensions or categories that influence directly the development of students’ competence: (i) Basic resources or knowledge base that involves basic definitions, facts, notations, formulae, algorithms and fundamental concepts associated with a particular area or theme, including ways to access that knowledge. (ii) Problem solving strategies that involve ways to represent and analyze the problems to understand and solve them. Some examples of these strategies are searching for subgoals; finding an easier or analogous problem; decomposing the problem, visualizing the problem using a diagram; working backward, etc. (iii) Metacognitive strategies that involve knowledge about one's own cognitive functioning (what do a I need and how I use that knowledge?) and strategies to monitor and control one's cognitive processes (what am I doing? Why am I doing it? Where am I going?). (iv) Beliefs and affective components that include students' conceptualization about mathematics and problem solving and students' attitudes and disposition to be engaged in mathematical activities. It is evident that these components are related and have been used widely to document and analyze students' problem solving behaviors. The dimensions or categories identified by Schoenfeld to characterize students and experts' ways of mathematical thinking emerge from observing and analyzing in detail how they solve a variety of problems. An example of the type of problems that Schoenfeld asked students or experts to solve was:

> Inscribe a square in a given triangle. Two vertices of the square should be on the base of the given triangle, the other two vertices of the square on the other two sides of the triangle, one on each (Schoenfeld, 1985, p. 85).



How to inscribe a square in a given triangle



The use of heuristic strategies

Relevant questions that subjects may formulate and pursue during the solution of this problem involve: What conditions or properties should a triangle hold to inscribe a square in it? Where to locate one of the upper vertices of that inscribed square? How can I inscribe a square in an isosceles or equilateral triangle? Can I draw a square having two vertices on one side of the given triangle? Etc.

The analysis of students competences based on the consideration of those dimensions has also detected flaws or misconceptions in students' competences and similar frameworks of analysis have been developed to explicate the origin and ways to examine students' mathematical behaviors. Specially Perkins and Simmons (1988) proposed to discuss students' problem solving performances through what they call frames of knowledge that include: The content frame (definitions, facts, and algorithms together with metacognitive strategies related to their use); the problem solving framework (specific and general problem solving strategies, beliefs and self-regulatory process); the epistemic framework (strategies and criteria to validate knowledge); and the inquiry framework (specific and general strategies to generate, criticize and extend knowledge). Thus, students' problem solving competences and flaws are explained in terms of the degree of robustness they have developed in their problem solving experiences. In particular, the lack of development of those frameworks provides useful information to explain the students' difficulties to deal with mathematical problems. For example, a student who responds that $(a+b)^2 = a^2 + b^2$ with $a,b \in R$ has not developed a means or criteria to validate this type of statement (epistemic framework).

Problem solving has been the focus of substantial research in mathematics education during the last three decades. This perspective has influenced notably distinct curriculum proposals (NCTM, 2000) that suggest not only a reorganization of curriculum contents in terms of lines of mathematical thinking (numbers and operations, algebra, patterns and functions, geometry and special relationships, measurement, and data analysis and probability) but also the development of mathematical processes or cognitive actions associated with the practice of the discipline (problem solving, reasoning and proof, communication, connections and representations).

Representations and Visualization. We take information from studies that recognize not only that representations and visualization play a crucial role in students' comprehension of mathematics but also that students' development of mathematical competences can be explained in terms of the use of various representations. We acknowledge that the term "representation" has been used in different domains, including psychology, philosophy, and education. The development of mathematical knowledge and learning can be traced in terms of the type of representations used to think of mathematics. "Much of the history of mathematics is about creating and refining representational systems, and much of the teaching of mathematics is about students learning to work with them and solve problems with them" (Lesh, Landau, & Hamilton, 1993, cited in Goldin & Shteingold, 2000, p. 4). As Thompson and Saldanha (2003, p. 98) stated:

> Mathematicians rely heavily on symbols systems to aid their reasoning. Symbol systems are tool for them. Mathematicians therefore strive to

> develop symbols systems (inscriptions and conventions for using them) that capture essential aspects of their intuitive understanding and means of operating, so they need not rely explicitly on conceptual imagery and operations as they move their reasoning forward or generate further insight.

Fey (1990) states that "in mathematics the representations become objects of study themselves –sources of new abstractions that, surprisingly often, serve as useful models of unanticipated patterns in concrete situations" (p. 73). Understanding mathematical ideas and solving mathematical problems are processes that involve the students' use of distinct types of representations. Indeed, Lesh, Behr, and Post (1987) state that "capitalizing on the strengths of a given representation is an important component of understanding mathematical ideas" (p. 87). For instance, when solving the quadratic equation, $4x^2 + 45x - 225 = 0$, using the general formula, the discriminant can be expressed as $45^2 + (16)(225)$. It also can be represented as $(5^2)(3^4) + (2^4)(3^2)(5^2)$; furthermore, as $(3^2)(5^2)(9+16)$ or as $(3^2)(5^4)$. This last representation leads to the solution immediately, that is $x = -15$ and $x = \frac{15}{4}$. Similarly, the quadratic function associated with that quadratic equation can be expressed as $y = ax^2 + bx + c \quad (a \neq 0)$, and this expression can be written as

$$\frac{y}{a} = x^2 + \frac{bx}{a} + \frac{c}{a} \Rightarrow \frac{y}{a} - \frac{c}{a} = x^2 + \frac{bx}{a} \Rightarrow \frac{y}{a} - \frac{c}{a} + \frac{b^2}{4a^2} = x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} \Rightarrow$$
$$\frac{y}{a} + \frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2 \Rightarrow \frac{y}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} \Rightarrow$$
$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \Rightarrow y = a\left[x - \left(\frac{-b}{2a}\right)\right]^2 + \frac{4ac - b^2}{4a},$$

and the latter form gives directly the vertex where the parabola reaches an extreme value (maximum or minimum) $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. In short, "writing $f(x) = x^2 - 1$ as $f(x) = (x+1)(x-1)$ tells you what inputs produce 0 as an output" (Cuoco, 2002, p. 296). That is, a proper representation of mathematical objects sheds lights on properties embedded in those objects.

In the same vein, Duval (1999) suggests that the use of several registers of representation (natural language, symbolic expressions, graphs, diagrams) of mathematical objects and the coordination of those registers are important in developing students' understanding of mathematics. Then it is natural to ask: What does "coordination" of those registers mean? What questions are important to ask in order to think and represent mathematical objects through different representations? How can that coordination among different registers be assessed? These questions seem to be the core or grounding basis of the representation and visualization framework.

Goldin and Shteingold (2001) suggest that mathematical representations are not isolated entities but part of systems that include a set of properties and rules (structure) needed to operate within and among those representations. "A specific formula, or equation, a concrete arrangement of base-ten blocks, or a particular graph in Cartesian coordinates makes sense only as part of a wider system within which meaning and conventions have been established" (p. 1). Thus, students' ways to represent and connect mathematical knowledge function as a vehicle to understand that knowledge deeply and use it in problem solving situations.

Kaput (1994) suggests that the use of calculators and computer technology can transform some traditional external representations from being static to function as dynamic configurations. In this context, there is an ongoing interaction between students' internal and external representations during their development of mathematical experiences. "An external character is experienced as meaningful or not, according to whether it matches the individual's internal representation of characters in a system that for him or her is operative" (Goldin & Kaput, 1996, p. 404). That is, "it is the internal level that largely determines the usefulness of such external representational systems, according to how the individual understands and interacts with them" (Goldin, 2002, p. 211).

Thus, focusing on students' construction of powerful representational systems becomes an important goal in mathematical instruction. Indeed, as Goldin (2002) mentioned:

> Sometimes one considers the external to represent the internal (e.g., when a student expresses a relationship he has in mind by drawing a graph). At other times, or even simultaneously, one can consider the internal to represent the external (e.g., when a student visualizes what is described by a graph or by an algebraic formula) (p. 211).

It is also important to recognize that students develop representational systems based on prior systems and over time. In this context, Goldin (2002) identifies three important stages that characterize the students' development of representational systems: (i) an inventive/semiotic stage, in which new internal configurations are constructed based on previously established representation; (b) a period of structural development; and (c) an autonomous stage, in which the new representational system functions flexibly and powerfully with new or more general meaning in new contexts.

There is a direct connection between problem solving and representation and visualization frameworks, since problem solving activities include special attention for students to analyze phenomena mathematically in terms of function graphs, algebraic and numerical representations between variables; flow charts, scale drawings and use of tables. Indeed, the use of representations permeates all problem-solving activities.

> Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understanding to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling (NCTM, 2000, p. 67).

Models and Modeling. We began by characterizing key elements associated with research that has been done under the umbrella of a "models and modeling perspective" (Lesh & Doerr, 2003). The term model is a key ingredient to explain students' learning of mathematics:

> A model is a system consisting of elements, relations among elements, operations that describe or explain how the elements interact, and the patterns or rules that apply to the preceding relations and operations…To be a model, a system must be used to describe, think about, interpret, explain, or make predictions about the behavior of some other phenomena or experienced system. A mathematically significant model must focus on the underlying structural characteristics of the experienced system (Doerr & Tripp, 2000, p. 231)

Thus, students, in their attempts to understand or solve problems, develop or construct mathematical models that help them reason about a system (explanatory power) and also can be used to arrive at new inferences or learn new content (predictive power). In this context, students develop models to construct, describe, or explain significant systems or phenomena they encounter in terms of mathematical resources or mathematically. Also, and more importantly, students develop conceptual systems and use them to build new concepts.

> Learning mathematics involves the development of models where the emphasis is on the underlying structural characteristics of the system and on the ability to reason with and about the system…The development of a new model is based on reasoning that draws on existing models that are related to the new problem situation in some way. The reasoning that occurs in an encounter with a problem situation may involve analogy from a familiar or at least partially understood system to a new system with an unfamiliar mathematical structure (Doerr & Tripp, 2000, p. 234)

This perspective recognizes the interaction and interdependences of mental or internal models (representations that are active while working on particular problem and guide the use of inferences and mental operations) and external models (those that are expressed by the use of different means: language, symbols, diagrams, or metaphors).

> Mismatches between a learner's interpretation and another's, as well as mismatches between one learner's interpretation and some external representation, can create the need for new interpretations or representations. This can lead to changes or shifts in thinking by one or more learners, resulting in a refined, potentially more powerful model (Doer & Tripp, 2000, p. 233).

In this perspective it is recognized that modeling activities are important for students to reveal their various ways of thinking and favor the development of their conceptual systems as a result of solving the activities. In brief:

> Modeling is seen as the interaction among three types of systems: (a) internal conceptual systems, (b) representational systems that function both as externalization of internal conceptual systems and as internalization of external systems, and (c) external systems that are experienced in nature or are artifacts that were constructed by others …we see these systems as overlapping, interdependent, and interacting. It is the interdependence and interactions that are foreground here and are central to our analysis of students learning from a modeling perspective (Doer & Tripp, 2000, p. 235.)

An important goal in a model-modeling environment is that students develop conceptual systems or models to make sense of math-rich problem solving situations. In this process, students need to express, test, revise, reject or construct their ideas.

> In model-eliciting activities, students produce conceptual tools that include explicit descriptive or explanatory systems that function as models which reveal important aspects about how students are interpreting the problem solving situations (Lesh & Doerr, p. 9).

Thus, mathematics knowledge that students display during their interaction with the task initially depends on what sense of the tasks they make. Since, students' problem solving mission is to develop a tool that can be useful or transferable in other situations, then students focus also on examining mathematical patterns and structure involved in their solution approaches. Students go beyond thinking with a model to also thinking about it. In this perspective, the task becomes a vehicle to access and extend students' mathematical knowledge.

> Thinking mathematically is about constructing, describing, and explaining at least as much as it is about computing, it is about quantities (and other mathematical objects) at least as much as it is about naked numbers; and it is about making (and making sense of) patterns and regularities in complex systems at least as much as it is about pieces of data. Also, relevant representation systems include a variety of written, spoken, constructed, or drawn media; and, representational fluency is at the heart of what it means to understand most mathematical constructs (Lesh & Doerr, p. 16).

It is noted that there are common grounds in both representations and visualization and models and modeling perspectives. In some statements, changing the world representations by models seems to make little difference to describe conceptual systems. In defending their terms choice, Lesh and Doerr (2003) argue "we have adopted simpler terminology that ordinary people consider to be productive and unpretentious – as long as these interpretations are close to those we intend, without carrying too much unintended conceptual baggage" (p. 8). However, a key aspect in a model and modeling perspective is the recognition that problem solutions, in general, involve several "modeling cycles" in which descriptions, explanations, and predictions are gradually refined, revised, or rejected–based on feedback from trial testing. In addition, during the students' interaction with the task, students themselves should be able to monitor their problem solving processes:

> [In problem solving] several levels and types of responses nearly always are possible (with one that is best depending on purposes and circumstances), and students themselves must be able to judge the relative usefulness of alternative ways of thinking. Otherwise, the problem solvers have no way to know that they must go beyond their initial primitive ways of thinking; and, they also have no way of judging the strengths and weaknesses of alternative ways of thinking–so that productive characteristics of alternative

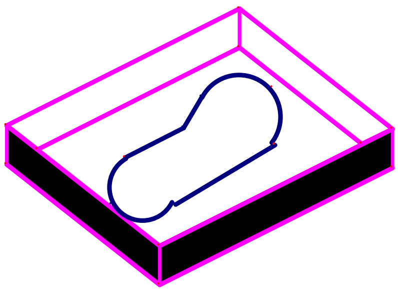
> ways of thinking can be sorted out and combined (Lesh & Doerr, 2003, p. 18).

Lester (2005) identifies key features associated with models and modeling perspectives:

> (a) [M]akes use of a variety of representational media to express the models that have been developed, (b) is directed toward solving problems (or making decisions) that lie outside the theories themselves (as a result, the criteria for success also lie outside the theories), (c) is situated (i.e., models are created for a specific purpose in a specific situation), and (d) the models are developed so that they are modifiable and adaptable (p. 460).

Designing a model-eliciting activity involves thinking of a situation in which students have the opportunity of developing and refining mathematical constructs in order to represent and examine relations associated with a task or problem. A good example is the Big Foot Problem that has been extensively used with middle school students. A fundamental mathematics theme guiding this activity is the use of proportional reasoning. The statement of the task is:

> Early this morning, the police discovered that, sometime late last night, some nice people rebuilt the old brick drinking fountain in the park where lots of neighborhood children like to play. The parents in the neighborhood would like to thank the people who did it. All the police could find were lots of footprints. One of the footprints is shown here. The person who made this footprint seems to be very big. But to find this person and his friends, it would help if we could figure out how big he is?----Your job is to make a "HOW TO" TOOL KIT that police can use to make good guesses about big people are -just looking at their footprints. Your tool kit should work for footprints like the one shown here. But it also should work for other footprints.



Footprint

It is up to researchers and practitioners to judge the pertinence of each framework in accordance with what better fits into their educational goals and interest. However, as Boaler, Ball, and Even (2003) stated:

> …[R]esearchers must develop a peculiar constellation of attitudes that include being skeptical, being open to surprise, trying to prove one's ideas wrong, and considering alternatives. …Reading widely and making good use of theory and ideas in one's own domain and others are other critical aspects of the research process (p. 493).

Results and Discussion

We organized our inquiry around themes that were taken as a reference to formulate questions that orient and guide the analysis. These themes were examined in order to identify main differences or contrasts among the perspectives. At the outset, it is convenient to identify the scope of each perspective. That is, it is important to recognize the focus and type of explanation favored or taken in each perspective to explicate students' mathematical behaviors. In particular, there is interest to discuss the students' processes of developing or constructing mathematical knowledge. Two main trends were identified: Perspectives or approaches that explain the development of students' mathematical competences in terms of discussing global tendencies based on the identification of resources, strategies, and metacognitive behaviors; and those that pay attention to the students' microscopic behaviors shown during the processes of understanding particular mathematical ideas. Thus, the scope provides the context to discuss other themes.

Scope of Each Perspective. There is evidence that **problem-solving** perspectives seem to provide a useful framework to analyze global aspects of students' mathematical competence. For example, problem solving research results often recognize the importance for students to conceptualize a vision of mathematics consistent with the practice of the discipline, to monitor their problem-solving processes and to develop the knowledge base or resources to comprehend and solve nonroutine problems. However, this perspective focuses mainly on explaining general students' problem solving behaviors rather than providing detailed explicit information about the students' development of proper mathematical problem solving competences. Problem solving dimensions (basic resources, cognitive and metacognitive strategies, beliefs and affective components) become important to characterize students' mathematical competences in general terms (the extent to which students exhibit them) but fall short in explaining ways in which students develop those dimensions in agreement with the practice of the discipline. It is also fair to mention that Schoenfeld's problem solving course taught at the University of

California, Berkeley has been an important indication to show the successful application of research results in instructional practices (Schoenfeld, 1998).

Representation and visualization perspectives focus on explicating micro-behaviors around the students' learning processes that involve description of ways that students transit from the use of one representation to another. Here it is recognized that representation and visualization plays a fundamental role in thinking and learning mathematics. In this framework, the use of semiotic systems and the understanding of how they function during the students' learning become important aspects to explain the process of learning and understanding mathematical concepts.

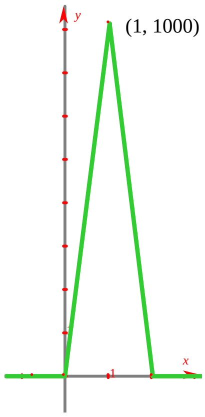
Models-modeling emphasizes the subjects' construction of conceptual systems (models) but offers little information regarding how students themselves develop new knowledge to construct more robust models. Indeed, there is no explicit information about the role of teachers in orienting students towards the construction of those models.

Regarding the type of mathematical vision endorsed by each perspective, it is possible to recognize features associated directly with the practice or development of mathematics with the problem solving perspective. That is, mathematics is seen as a science of patterns that is developed or learned within an environment that favors and encourages processes of inquiry or reflection that lead to the understanding of phenomena through the use of mathematical resources. Models-modeling perspectives see the discipline as a system of relationships that can be expressed through models. Thus, conceptual systems, cognitive systems and models are fundamental ingredients to explain students' processes of understanding mathematical ideas. Representation perspectives identify mathematics as semiotic representation system that deals with mathematical ideas and their transformations based on the use of different registers of representations.

The Role of Problems. What types of tasks or problems are used, within the framework, to explore, promote, and document students' learning? This question becomes important to analyze purposes and ways to use the problems during the research. "…[I]t is so important to justify the choice of the mathematical tasks used in a research, not just in terms of the general goals and theoretical framework of the research, but in terms of the specific characteristics of the task" (Sierpinska, 2004, p. 10). In this context, problem-solving perspectives recognize nonroutine tasks as a vehicle for students to exhibit their ways of thinking and problem solving behaviors. These problems are embedded in diverse contexts and their understanding require the use of resources and strategies that may lead students to solve them and eventually to pose new questions or problems. In particular, nonroutine problems provide opportunities for students to explore distinct ways of solution, to use diverse representations, to formulate conjectures, to present arguments and to communicate results (Schoenfeld, 2002). An example of non-routine problem to explore students' ways of reasoning is:

Give an example of a function whose domain is the real numbers, it is continuous and nonnegative, with a maximum value of 1000 at x =1 and whose area under its graph is smaller than 1/1000.

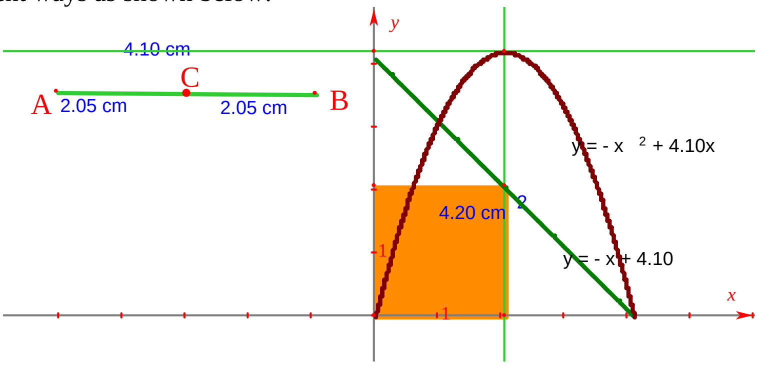
Here the subjects have the opportunity to ask for example, what does it mean for a function be continuous and nonnegative? Can the value of the function be zero and be called nonnegative? Etc. to eventually examine function's candidates like the figure:



The problems or activities used in models-modeling involve open-ended tasks, presented in realistic and meaningful contexts for students. "The activities are inbuilt with ways for students to realistically assess the quality of their own ways of thinking without predetermining what their final solution should look like" (Lesh & Yoon, 2004). Indeed, a particular feature of this type of tasks is that there is no one particular solution that all students need to achieve, rather plausible solution models depend on the set of conditions that students judge to be important to consider while approaching the task. An example of a thought revealing activity is:

> The paper airplane problem in which students are asked to read an article about how to make a variety of different types of paper airplanes. Later students make their own paper airplanes and test their flight characteristics by trying to hit a target following different kinds of flight paths. In each test, students take measures of (a) total distance flown, (b) the distance from the target, and (c) the time in the flight. The mission for students is to write a letter to students in another class describing how such data can be used to assess paper airplanes for following four kinds of characteristics: (a) best floater (going slowly for a long time), (b) most accurate, (c) best boomerang, and (d) best overall.

Representation perspectives rely on the use of problems in which students can use multiple representations to discuss relationships and mathematical properties. Thus, numeric, geometric representations often are relevant to analyze and discuss phenomena or problems mathematically. In addition, problems or phenomena that can be analyzed in terms of algebraic, numeric, or graphic representations become important in this perspective to grasp key mathematical concepts associated with the situation in study. The problem: "Given two real numbers with a fixed sum P, show when the product of those numbers is maximum" can be represented in different ways as shown below:



A problem with multiple representations (How are they related?).

In order to identify relevant features associated with each perspective, we elaborate on themes that define our inquiry framework (sketched previously) to characterize aspects related to the discipline (how mathematics is characterized), the type of tasks (what types of tasks promote mathematical learning); the processes of learning (how learning takes place) and evaluation (how mathematics competence is assessed). The information presented next shows main differences in terms of the use of language (terminology) and foci around each perspective. Thus, Table 1 provides useful information to contrast common themes in each perspective.

An important aspect that is not addressed directly in each perspective is the systematic use of computational tools in the students' processes of learning or comprehending mathematics or solving problems. That is, we observe that main research results associated with each perspective come from examining students' work that involves the use of paper and pencil (Schoenfeld, 1992; Duval, 1999) and a few cases from students using Excel to work on thought-revealing activities (Lesh & Doerr, 2001). In this context, it becomes important to reflect on the extent to which the systematic use of computational tools in learning activities asks for a re-examination of basic principles associated with a particular framework in order to explain the development of students' mathematical competences. We recognize that different computational tools may offer distinct possibilities for students to interact with mathematics tasks. For example the use of dynamic geometry software

Table 1

Contrast Common Themes in Different Perspectives

Perspectives	Mathematics Features	Type of Tasks	Processes of Learning	Learning Environments	Evaluation
Problem-Solving	Mathematics as a science of patterns Direct relationships between mathematical practice and students learning. Mathematical thinking involves the formulation of questions, conjectures, relationships, and the use of distinct types of arguments.	Non-routine tasks that include problems to be solved during the class time, homework problems and projects. Transforming routine tasks into nonroutine activities through processes that involve formulation of questions.	Problem solving dimensions: Basic resources, cognitive and metacognitive (monitoring and self-control) strategies, beliefs' systems (affect).	Classroom as a mathematical microcosm. Classroom as mathematical communities. Students work in small groups, whole group participation The instructor as a scaffolding	Solution processes of nonroutine problems. Students competences in mathematical processes that involve: Representations Communication Conjecturing Formulation of questions Distinct types of arguments Monitoring
Representation	Distinction between mathematical objects and their representations. Mathematical thinking expressed through systems of semiotic representations.	Tasks that involved the use of multiple representations.	Coordination of representations Transit from one representation to the other (meaning). Operations within the same register; conversion of registers; and discrimination of registers.	Problem solving environments to promote students' construction of representations of mathematical ideas and their connections.	Evidence that students display connections between registers. Recognition of the same object through different representations.
Models-Modeling	Mathematics as a system of relationships useful to understand and make sense of distinct phenomena. Solving a situation or task leads to the construction of tools for thinking. Mathematics is seen as a system with elements, operations, rules, and relationships.	Tasks embedded into distinct contexts. Solutions involve explanations, descriptions, interpretations, representations, operations, algorithms, arguments, extensions, revisions, adjustments, etc.	Learning involves the construction of models or conceptual systems. Learning is expressed through a sequence of modeling cycles that might evolve from being non-stables models to robust and stables models.	Learning environments are designed around the discussion of model-eliciting tasks. Students often work in pair of groups of three and the teacher functions as a monitor during the sessions.	Students' development of conceptual tools to be used in solving family of problems. Student-self evaluation: The student becomes "the client" who reviews and assesses his/her results and those of others.

may become an important tool to identify invariants or relationships associated with particular problem of phenomenon by constructing a dynamic representation of the problem, while the use of Excel may result a useful tool to represent relevant information of the problem and relationships in tables to detect patterns or visual behaviors. In this context, it becomes important to examine the extent to which the questions that students ask, the representation they utilize and the arguments they use to support their results with the use of technology are consistent with those that appear in paper and pencil approaches.

> Regardless of the particular tools that are used, they are likely to shape the way we think. Mathematical activity requires the use of tools, and the tools we use influence the way we think about the activity…[Understanding] is made up of many connections or relationships. Some tools help students make certain connections; other tools encourage different connection (Hiebert et al., 1997, p. 10).

Our position is that more data need to be generated and analyzed to actually characterize the type of mathematical thinking that emerges when students systematically use technology in their processes of understanding mathematical ideas. As a consequence, frameworks that explain students' mathematical competences need to be adjusted constantly in accordance with what students develop in their problem solving approaches that incorporate the use of distinct technological tools. In addition, it is important to revise different theoretical positions to identify common ground and ways to reconcile and unify main principles rather that rejecting them without presenting solid arguments. As Goldin (2003) stated:

> …the theme of reconciling and unifying diverse theoretical perspectives, and obstacle to progress has been the dismissal by prevailing belief systems of important constructs from other systems, on a priori (but unscientific) grounds. These dismissals, when taken seriously, have had damaging consequences for educational practice in mathematics (p. 282).

Goldin went on to provide several examples to illustrate his point. Regarding the position of some cognitive scientists, he stated:

> …[For example], other cognitive theorists seem to believe that all thought – and, in particular mathematical thought- consists exclusively of metaphors of various sorts…If this view is taken seriously, we are likely to see a further devaluation and discrediting of formal systems and

> abstract mathematics as classroom topics, again with unfortunate consequences. Furthermore, the prevailing cognitive theories of mathematics education have placed little emphasis on affect (p. 282).

Thus, it is important for the mathematics education community to communicate and discuss ideas around theoretical and practical work in order to value and appreciate its potential since "…the lack of communication entails the impossibility of accumulating and the habit of "reinventing the wheel" (Sfard, 2005, p. 399).

Remarks

What features are relevant in mathematics knowledge? How do students learn new mathematical knowledge? How can students construct or develop mathematical concepts beyond those they have learned? What processes entail the students' abilities to articulate their competence in learning mathematics? What type of tasks becomes relevant for students to develop mathematical thinking? What instructional conditions are important for students to learn? These types of questions were relevant and part of an inquiry framework to examine the scope and explanatory power associated which each perspective. The extent to which each perspective addresses explicitly themes involved in these types of questions became important during the analysis of the sources related to each perspective. Indeed, issues that emerged during the analysis and were important to structure and ponder the information that was analyzed involved:

(i) **Scope of the framework**, here it was evident that problem-solving and models-modeling perspectives focus on explicating general or macro cognitive processes around students' learning; while the representation perspective seems to focus on explaining particular and detailed students' learning behaviors.

(ii) **Sources and ways to inform**, that is, the nature itself of each perspective differs since the problem-solving perspective, for example, addresses directly issues related to mathematics knowledge (what is mathematics?) and ways to create a mathematical microcosm in classroom, while the representation perspective only deals with these components implicitly. Models and Modeling perspective seems to emphasize the importance for students to apply mathematical resources to understand and solve problems that initially are meaningful to them or their environment.

(iii) **The need for an evaluation tool**, an initial difficulty arose when deciding what to look for in each perspective, that is, the existence of distinct perspectives in the field requires the development of ways to examine and contrast their fundamental principles and tenets, in terms of evaluating the type of contributions to understanding relevant problems of the discipline. The questions that helped frame the discussion represent an initial point to think of that evaluation tool.

(iv) **Formulation of questions**, a common feature associated with the three perspectives is that students develop, construct and transform their own understanding as a result of posing relevant questions and pursuing them through different means and constantly revising them within a learning community.

(v) **Technology and the perspectives**. The use of technology has influenced notably the ways that students represent and examine mathematics knowledge, and frameworks needs to re-adjust their principles in accordance with the types of transformations that are produced by the use of technological artifacts in students' learning.

Finally, we have sketched features of an inquiry framework to identify and discuss elements that support some research perspectives in the discipline. The existence of a variety of perspectives to frame teaching and learning studies makes necessary and relevant for researchers, teachers and students to evaluate the potential in using those frameworks. Thus, the questions that we have proposed and used to delve into the frameworks need to be examined and refined in order to go further on what is important for the framework to include.

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References

Arcavi, A. (2000). Problem-driven research in mathematics education. *Journal of Mathematical Behavior*, *19*, 141-173.

Boaler, J. Ball, D. L. & Even, R. (2003). Preparing mathematics education researchers for disciplined inquiry: Learning from, in, and for practice. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 491-521). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the twenty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Vol. 1 (pp. 3-26). Columbus, HO: ERIC Clearinghouse for Science, Mathematics, and Environment Education.

ESM Editors (2002). Reflection on educational studies in mathematics: The rise of reseach in mathematics education. *Educational Studies in Mathematics*, *50*(3), 251-257.

Fey, J. T. (1990). Quantity. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy*. Washington, DC: National Academic Press.

Goldin, G. & Kaput, J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397-430). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Goldin, G. (2002). Representation in mathematical learning and problem solving. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 197-218). Mahwah, NJ: Lawrence Erlbaum Associates.

Goldin, G. (2003). Representations in school mathematics: A unifying research perspective. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 275-285). Reston, VA: National Council of Teachers of Mathematics.

Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (2001 Yearbook) (pp. 1-23). Reston, VA: National Council of Teachers of Mathematics.

Greeno, J. G., Collins, A., M., & Resnick, L. B. (1996). Cognition and learning. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 15-46). New York: Macmillan.

Heid, M. K, Middleton, J., Larson, M., Gutstein, E., Fey, J. T., King, K., Strutchens, M. E., & Tunis, H. (2006). The challenge of linking research and practice. *Journal for Research in Mathematics Education*, *37*(2), 76-86.

Hiebert, J. Carpenter, T. P., Fennema, E., Fuson, K.C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). Making sense. *Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Kaput, J. J. (1994). The representational roles of technology in connecting mathematics with authentic experience. In R. Biehler, R. W. Scholz, R. Strässer, & B. Winkelman (Eds.), *Didactic of mathematics as a scientific discipline* (pp. 379-397). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Lesh, R. & Doerr, H. M. (2003) (Eds). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Lesh, R. & Doerr, H., M. (2003). Foundation of a models and modeling perspective on mathematics teaching learning, and problem solving. In R. Lesh, & H. M. Doerr (Eds.), *Beyond constructivism. Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3-33). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R. & Yoon, C. (2004). Evolving communities of mind –in which development involves several interacting and simultaneously developing strands. *Mathematical Thinking and Learning, 6*(2), 205-226.

Lester, F. K. & Kehle, P. E. (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modelling perspectives on mathematics problem solving, learning, and teaching* (pp. 501-517). Mahawah, NJ: Lawrece Erlbaum Associates.

Lester, F. K. (2005). On the theoretical, conceptual, and philosophical foundation for research in mathematics education. *ZDM* 2005, *37*(6), 457-467.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Perkins, D. N. & Simmons, R. (1988). Patterns of misunderstanding: An integrative model for science, math, and programming. *Review of Educational Research, 58*, 303-326.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. H. (1998). Reflections on a course in mathematical problem solving. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 81-113). Washinton, DC: American Mathematical Society.

Schoenfeld, A. H. (2000). Purpose and methods of research in mathematics education. *Notices of the AMS, 6*, 641-649.

Sfard, A. (2003). Balancing the unbalanceable: The NCTM Standards in the light of theories of learning mathematics. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 353-392). Reston, VA: National Council of Teachers of Mathematics.

Sfard, A. (2005). What could be more practical than good research? *Educational Studies in Mathematics*, *58*, 393-413.
Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the learning of mathematics*, *24*(2), 7-15.

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